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Electromagnetic energy–momentum tensors in general dispersive bianisotropic media

RUO-YANG ZHANG,[1,*] ZHONGFEI XIONG,[2] NENG WANG,[3] YUNTIAN CHEN,[2,4] AND C. T. CHAN[1]

[1] Department of Physics, The Hong Kong University of Science and Technology, Hong Kong, China
[2] School of Optical and Electronic Information, Huazhong University of Science and Technology, Wuhan, 430074, China
[3] Institute of Microscale Optoelectronics, Shenzhen University, Shenzhen, 518052, China
[4] Wuhan National Laboratory of Optoelectronics, Huazhong University of Science and Technology, Wuhan, 430074, China
*Corresponding author: ruoyangzhang@ust.hk

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The macroscopic electromagnetic (EM) energy–momentum tensor is one of the most important quantities characterizing the propagation and interaction of light in materials. In recent years, while exotic optical effects in various kinds of bianisotropic materials have been discovered, there still lacks a rigorous analysis of the energy and momentum of EM fields in such general cases. In this paper, using Noether's theorem and the "Abrahamization" procedure, we obtain generalized Minkowski and Abraham EM energy–momentum tensors, applicable for both arbitrary time-dependent real EM fields and complex-valued analytic signals, in generic lossless bianisotropic media with frequency dispersion. The frequency dispersion of the materials modifies the expressions of EM energy density and Minkowski momentum, making them different from their familiar forms in nondispersive media. Our results reveal that the generalized Minkowski momenta for both real fields and analytic signals are conserved in source-free homogeneous media, while the Abraham momenta, characterizing the centroid motion of light, can change over time, which leads to the counterintuitive phenomenon that wave packets can travel along curved trajectories even in homogeneous bianisotropic media. We also show that the energy–momentum tensor for analytic signals derived from the action principle directly gives the conservation law of time-averaged fields and hence can describe the envelope evolution of waves in quasi-monochromatic approximation.

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1. INTRODUCTION

Energy and momentum are fundamental properties of light and are at the heart of the study of optical propagation, structured optical fields, and light–matter interaction. However, there has been a century-long debate about the definitions of the electromagnetic (EM) energy–momentum (E-M) tensor in macroscopic media [1–14]. Among various proposed formulations of EM E-M tensors [9], the two most famous definitions are the Minkowski tensor and the Abraham tensor. The Minkowski E-M tensor is derived from Noether's theorem associated with spacetime translation symmetry [11], so it can be regarded as the canonical momentum of photons [4–6]. In contrast, the Abraham momentum is proportional to the Poynting vector and is suggested to be kinetic momentum [4–6]. In previous studies of E-M tensors, researchers focused on nondispersive anisotropic materials [3,11] or generalized the concept to dispersive isotropic materials for either time-averaged monochromatic waves [12–14] or general instantaneous fields [1,2,6,10]. On the other hand, profiting from the substantial progress of metamaterials, the extended class of optical media, so-called bianisotropic materials, has attracted more and more attention. Many optical concepts have been generalized for bianisotropic media [15–19]. Numerous intriguing optical phenomena in such media, such as topological effects [20–22] and synthetic gauge field for light [23,24], have been discovered. However, as basic properties of EM fields, the expressions of Minkowski and Abraham tensors of light have never been generalized for this most general class of media.

It is a common belief that light propagates in a straight line in homogeneous media, as a consequence of momentum conservation. From an intuitive physical picture, the traveling direction of light should be parallel to its momentum, and the conservation of EM momentum leads to a straight trajectory of centroid. Specious bending of light in vacuum, so-called "accelerating waves," was proposed using specific optical beams, e.g., Airy beams, whose main intensity maxima trace a curved trajectory [25,26]. Nevertheless, the intensity centroid of an accelerating beam still travels in a straight line, as shown in Fig. 1(a). Very recently, the authors of the paper discovered that in a wide group of bianisotropic media (including usual biaxial dielectrics), the centroid of a Gaussian-type beam can propagate in a wavy trajectory even if the medium is homogeneous [24], as shown in

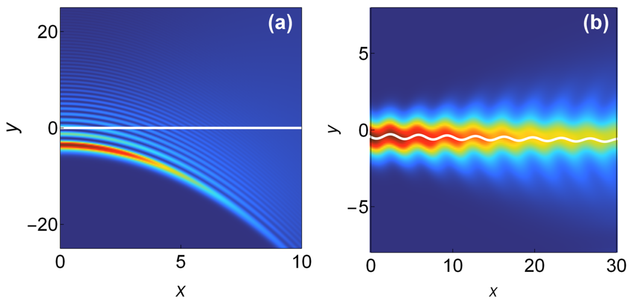

Fig. 1. Bending light in homogeneous media. (a) Accelerating Airy beam in vacuum. The main peak of the beam forms a parabola, and its centroid travels in a straight line [25]. (b) Zitterbewegung effect of light in a homogeneous "non-Abelian" anisotropic medium, where the centroid of a Gaussian-type beam moves along a wavy trajectory [24]. The white lines denote the centroid trajectories of the beams.

Fig. 1(b). This counterintuitive effect, called Zitterbewegung of light, apparently contradicts the belief of rectilinear propagation. Although the effect had been successfully explained based on a synthetic non-Abelian Lorentz force acting on the light induced by the material anisotropy [24], a more rigorous analysis on the exact conserved momentum in general bianisotropic media, and on the relation between the momentum and centroid motion of a wave packet, is highly desirable.

In this paper, we will fill these gaps via giving the E-M conservation law in generic lossless bianisotropic media with frequency dispersion. In the studies of EM energy and momentum in materials, there are two different major focuses. One is on the interactions between light and matter microscopically, such as the optical radiation pressure and stress inside materials [27–31], while the other one is on the macroscopic effect of light, regarding media as a non-dynamic background [10–13]. The interest of this work strictly follows the second approach. Following the method of power series expansion of the dispersive constitutive tensors in Ref. [10], we first generalize the traditional EM Lagrangian to a generalized one for EM fields in dispersive bianisotropic media involving infinite orders of derivatives. Then we derive the expression of the generalized Minkowski E-M tensor associated with spacetime translation symmetry via generalizing the variational formulation with higher-order derivatives. Based on an "Abrahamizatoin" procedure, we also derive the Abraham tensor in the general case. In addition to the real-valued instantaneous EM fields, we further obtain an independent E-M balance equation and the conserved tensor for complex-valued analytic signals [32,33]. We show that the analytic signal gives exactly the time-period-averaged fields for quasi-monochromatic waves. Therefore, the E-M balance equation of analytic signals depicts the motion of the envelope of wave packets. Then we demonstrate that the curved motion of the intensity centroid in the Zitterbewegung effect is directly governed by the Abraham momentum whose magnitude and direction can vary with time, while the Minkowski momentum is always conserved in homogeneous media. Our comprehensive study of the EM E-M tensors offers a new perspective in understanding the optical effects in general bianisotropic media.

2. CONSTITUTIVE RELATIONS IN DISPERSIVE BIANSITORPIC MEDIA

A. 3D Formulations for Real Vector Fields

For an arbitrary group of time-dependent EM fields (including sources) satisfying Maxwell's equations, the Fourier transforms between time and frequency domains are expressed as

$$\mathbf{X}(\mathbf{r}, t) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega \mathbf{X}_\omega(\mathbf{r}) e^{-i\omega t}, \quad \mathbf{X} \in \{\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{j}, \rho\}. \tag{1}$$

Since EM fields are real-valued functions in the time domain, their Fourier components of positive and negative frequencies respect

$$\mathbf{X}_{-\omega}(\mathbf{r}) = \mathbf{X}_\omega(\mathbf{r})^*. \tag{2}$$

In a generic dispersive bianisotropic medium, the optical properties of the medium are characterized by the constitutive relation in the frequency domain:

$$\begin{pmatrix} \mathbf{D}_\omega(\mathbf{r}) \\ \mathbf{B}_\omega(\mathbf{r}) \end{pmatrix} = \underbrace{\begin{pmatrix} \overleftrightarrow{\varepsilon}(\mathbf{r}, \omega) & \overleftrightarrow{\chi}_{\mathrm{em}}(\mathbf{r}, \omega) \\ \overleftrightarrow{\chi}_{\mathrm{me}}(\mathbf{r}, \omega) & \overleftrightarrow{\mu}(\mathbf{r}, \omega) \end{pmatrix}}_{\overleftrightarrow{M}(\mathbf{r},\omega)} \begin{pmatrix} \mathbf{E}_\omega(\mathbf{r}) \\ \mathbf{H}_\omega(\mathbf{r}) \end{pmatrix}, \tag{3}$$

where the constitutive tensor $\overleftrightarrow{M}(\mathbf{r}, \omega)$ is a function of both frequency and spatial coordinates, but we suppose that it is invariant with time. Furthermore, we confine our attention to nondissipative and passive media, which requires the constitutive tensor to be Hermitian, $\overleftrightarrow{M}(\mathbf{r}, \omega) = \overleftrightarrow{M}(\mathbf{r}, \omega)^\dagger$, or, equivalently [16],

$$\overleftrightarrow{\varepsilon}(\mathbf{r}, \omega) = \overleftrightarrow{\varepsilon}(\mathbf{r}, \omega)^\dagger, \quad \overleftrightarrow{\mu}(\mathbf{r}, \omega) = \overleftrightarrow{\mu}(\mathbf{r}, \omega)^\dagger,$$
$$\overleftrightarrow{\chi}_{\mathrm{em}}(\mathbf{r}, \omega) = \overleftrightarrow{\chi}_{\mathrm{me}}(\mathbf{r}, \omega)^\dagger = \overleftrightarrow{\chi}(\mathbf{r}, \omega). \tag{4}$$

One may argue that the lossless assumption is not consistent with the existence of frequency dispersion of the material, regarding the Kramers–Kronig relations between the real and imaginary parts of the constitutive tensors as a result of causality [33]. Nevertheless, this issue can be resolved, if the problem of concern focuses on a limited frequency range, where the material dispersion is strong but dissipation is negligible. For instance, according to the causal Drude and Lorentz models, the non-Hermitian imaginary part of permittivity tends to some discrete delta functions diverging at the resonant frequencies in the lossless limit [34,35]. It is hence safe to ignore the loss, provided that the frequencies of interest are away from resonances. We will see that the lossless condition (4) is crucial in establishing the Lagrangian description for EM fields.

To connect the 3D and 4D formulations of EM fields in the next section, we rewrite the constitutive equation as [16]

$$\begin{pmatrix}\mathbf{D}_\omega(\mathbf{r})\\ \mathbf{H}_\omega(\mathbf{r})\end{pmatrix}=\begin{pmatrix}\overleftrightarrow{\varepsilon} & \overleftrightarrow{\chi}\\ 0 & \overleftrightarrow{I}\end{pmatrix}\begin{pmatrix}\mathbf{E}_\omega(\mathbf{r})\\ \mathbf{H}_\omega(\mathbf{r})\end{pmatrix}$$

$$=\begin{pmatrix}\overleftrightarrow{\varepsilon} & \overleftrightarrow{\chi}\\ 0 & \overleftrightarrow{I}\end{pmatrix}\begin{pmatrix}\overleftrightarrow{I} & 0\\ \overleftrightarrow{\chi}^\dagger & \overleftrightarrow{\mu}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{E}_\omega(\mathbf{r})\\ \mathbf{B}_\omega(\mathbf{r})\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}\tilde{\overleftrightarrow{\varepsilon}}(\mathbf{r},\omega) & \tilde{\overleftrightarrow{\chi}}(\mathbf{r},\omega)\\ -\tilde{\overleftrightarrow{\chi}}(\mathbf{r},\omega)^\dagger & \overleftrightarrow{\mu}^{-1}(\mathbf{r},\omega)\end{pmatrix}}_{\overleftrightarrow{N}(\mathbf{r},\omega)}\begin{pmatrix}\mathbf{E}_\omega(\mathbf{r})\\ \mathbf{B}_\omega(\mathbf{r})\end{pmatrix}, \quad (5)$$

where $\tilde{\overleftrightarrow{\varepsilon}}=\overleftrightarrow{\varepsilon}-\overleftrightarrow{\chi}\cdot\overleftrightarrow{\mu}^{-1}\cdot\overleftrightarrow{\chi}^\dagger$, $\tilde{\overleftrightarrow{\chi}}=\overleftrightarrow{\chi}\cdot\overleftrightarrow{\mu}^{-1}$, and $\overleftrightarrow{I}$ denotes a 3D unit tensor.

As a result of Eq. (2), the constitutive tensors should respect

$$\overleftrightarrow{M}(\mathbf{r},-\omega)=\overleftrightarrow{M}(\mathbf{r},\omega)^*,\quad \overleftrightarrow{N}(\mathbf{r},-\omega)=\overleftrightarrow{N}(\mathbf{r},\omega)^*. \quad (6)$$

Therefore, even if $\overleftrightarrow{M}(\omega)$ is assumed to be frequency independent when $\omega>0$, the negative-frequency component has to be different from the positive-frequency one, unless $\overleftrightarrow{M}$ is purely real. Consequently, a complex-valued constitutive tensor $\overleftrightarrow{M}(\omega)$ is generically dispersive, which impels us to take account of frequency dispersion when deriving the EM E-M tensor as long as the medium is complex-valued. We further assume that the constitutive tensors can be expanded as power series [10]

$$\overleftrightarrow{M}(\mathbf{r},\omega)=\sum_{n=0}^{\infty}\overleftrightarrow{M}_n(\mathbf{r})\omega^n,\quad \overleftrightarrow{N}(\mathbf{r},\omega)=\sum_{n=0}^{\infty}\overleftrightarrow{N}_n(\mathbf{r})\omega^n, \quad (7)$$

with $\overleftrightarrow{M}_n=(-1)^n\overleftrightarrow{M}_n^*$ and $\overleftrightarrow{N}_n=(-1)^n\overleftrightarrow{N}_n^*$ according to Eq. (6). Then the constitutive relations in the time domain can be expressed by the compact forms

$$\begin{pmatrix}\mathbf{D}(\mathbf{r},t)\\ \mathbf{B}(\mathbf{r},t)\end{pmatrix}=\overleftrightarrow{M}(\mathbf{r},i\partial_t)\begin{pmatrix}\mathbf{E}(\mathbf{r},t)\\ \mathbf{H}(\mathbf{r},t)\end{pmatrix}, \quad (8)$$

$$\begin{pmatrix}\mathbf{D}(\mathbf{r},t)\\ \mathbf{H}(\mathbf{r},t)\end{pmatrix}=\overleftrightarrow{N}(\mathbf{r},i\partial_t)\begin{pmatrix}\mathbf{E}(\mathbf{r},t)\\ \mathbf{B}(\mathbf{r},t)\end{pmatrix}, \quad (9)$$

where the dispersive constitutive tensors convert to the functions of a time derivative operator.

B. 4D Formulations

The EM fields can be alternatively depicted by the four-vector potential $(A_\mu)=(-\varphi, c\mathbf{A})$, ($\varphi$ and $\mathbf{A}$ are the EM scalar potential and 3D vector potential, respectively), the 4D EM tensor $F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu$, and the 4D excitation tensor $G^{\mu\nu}$, where the Greek letters $\mu,\nu\in\{0,1,2,3,4\}$ denote the indices of 4D Minkowski coordinates. The correspondence between the components of 3D EM fields and 4D EM tensors is given by

$$F_{0i}=-F_{i0}=E_i,\quad F_{ij}=-F_{ji}=-c\epsilon_{ijk}B^k,$$

$$G^{0i}=-G^{i0}=-cD^i,\quad G^{ij}=-G^{ji}=-\epsilon^{ijk}H_k, \quad (10)$$

where the Latin letters $i,j,k\in\{i,2,3\}$ denote the indices of spatial coordinates, ϵ_{ijk} is the Levi–Civita symbol, and the Einstein summation convention of repeated indices has been adopted. Then the 4D expression of the constitutive relation in a generic bianisotropic medium can be written as

$$G^{\mu\nu}(x^\mu)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{2}C^{\mu\nu\alpha\beta}(x^i,k_0)(F_\omega)_{\alpha\beta}(x^i)e^{-ik_0x^0}\mathrm{d}k_0$$

$$=\frac{1}{2}C^{\mu\nu\alpha\beta}(x^i,i\partial_0)F_{\alpha\beta}(x^\mu), \quad (11)$$

where $k_0=\omega/c$, $\{x^\mu\}=\{ct,\mathbf{r}\}$, $(F_\omega)_{\alpha\beta}$ is the Fourier components of $F_{\alpha\beta}$ at the frequency $\omega=ck_0$, and $C^{\mu\nu\alpha\beta}$ represents the 4D constitutive tensor. In principle, the 4D covariant constitutive relation is valid in any reference systems. However, for the ease of dealing with dispersive media, we always assume that the background medium is temporally homogeneous in the selected reference, and hence the 4D constitutive tensor can be expressed as a function of spatial coordinates and frequency:

$$C^{\mu\nu\alpha\beta}(x^i,k_0)=\sum_{n=0}^{\infty}C_n^{\mu\nu\alpha\beta}(x^i)k_0^n. \quad (12)$$

Since both $F_{\alpha\beta}$ and $G^{\mu\nu}$ are antisymmetric tensors, we require

$$C^{\mu\nu\alpha\beta}=-C^{\nu\mu\alpha\beta}=-C^{\mu\nu\beta\alpha}, \quad (13)$$

which is consistent with Eq. (11) and can uniquely determine the 4D constitutive tensor by the 3D ones as follows [15,16]:

$$\begin{cases}C^{0i0j}=-c(\varepsilon^{ij}-\chi^{ik}\mu^{-1}{}_{kl}\chi^{\dagger lj})=-c\tilde{\varepsilon}^{ij},\\ C^{0ijk}=\chi^{il}\mu^{-1}{}_{lm}\epsilon^{jkm}=\tilde{\chi}_l^i\epsilon^{jkm},\\ C^{ijkl}=\frac{1}{c}\epsilon^{ijm}\epsilon^{kln}\mu^{-1}{}_{mn}.\end{cases} \quad (14)$$

The lossless condition Eq. (4) leads to the constraint

$$C^{\mu\nu\alpha\beta}(x^i,k_0)=C^{\alpha\beta\mu\nu}(x^i,k_0)^*$$
$$\Updownarrow$$
$$C^{\mu\nu\alpha\beta}(x^i,i\partial_0)=C^{\alpha\beta\mu\nu}(x^i,-i\partial_0)^*. \quad (15)$$

The correspondence between positive and negative frequencies Eq. (6) is also inherited by the 4D tensor

$$C^{\mu\nu\alpha\beta}(x^i,k_0)=C^{\mu\nu\alpha\beta}(x^i,-k_0)^*$$
$$\Updownarrow$$
$$C^{\mu\nu\alpha\beta}(x^i,i\partial_0)=C^{\mu\nu\alpha\beta}(x^i,i\partial_0)^*. \quad (16)$$

As we will see later, it is quite convenient to derive the E-M conservation law using the 4D covariant formulation of EM fields.

C. Complex Field Formulations—Analytic Signals

The condition $\mathbf{X}_{-\omega}=\mathbf{X}_\omega^*$ implies that the negative-frequency components of the real-valued fields are superfluous; thus, we can introduce a group of complex-valued fields by throwing away all the negative-frequency components:

$$\tilde{\mathbf{X}}(\mathbf{r}, t) = \int_0^\infty d\omega \mathbf{X}_\omega(\mathbf{r}) e^{-i\omega t}$$
$$= \mathbf{X}(\mathbf{r}, t) + i\mathbf{X}^{(I)}(\mathbf{r}, t),$$
$$(\mathbf{X} \in \{\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{j}, \rho, F_{\mu\nu}, G^{\mu\nu}\}), \quad (17)$$

which are the so-called *analytic signals* associated with the original real-valued EM fields [32,33]. Their imaginary parts can be obtained from the Hilbert transforms $\mathbb{H}[\mathbf{X}(\mathbf{r}, \cdot)](t)$ of the real fields $\mathbf{X}(\mathbf{r}, t)$:

$$\mathbf{X}^{(I)}(\mathbf{r}, t) = \mathbb{H}[\mathbf{X}(\mathbf{r}, \cdot)](t) = \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{\infty} \frac{\mathbf{X}(\mathbf{r}, t')}{t' - t} \mathrm{d}t', \quad (18)$$

where P. V. denotes Cauchy principal value. The Bianchi identity $\varepsilon^{\mu\nu\alpha\beta}\partial_\nu F^{(I)}_{\alpha\beta} = 0$ always holds. Also, the Hilbert transform of the Maxwell's equations shows that

$$0 = \mathbb{H}\left[\partial_\mu G^{\mu\nu} - J^\mu\right] = \partial_\mu \mathbb{H}\left[G^{\mu\nu}\right] - \mathbb{H}\left[J^\nu\right] = \partial_\mu G^{(I)\mu\nu} - J^{(I)\nu}, \quad (19)$$

where the property of Hilbert transform $\mathbb{H}[df(t)/dt] = d\mathbb{H}[f]/dt$ has been used. Therefore, $\{F^{(I)}_{\mu\nu}, G^{(I)\mu\nu}, J^{(I)\mu}\}$ is also a solution to Maxwell's equations, and so is the group of analytic signals $\{\tilde{F}_{\mu\nu}, \tilde{G}^{\mu\nu}, \tilde{J}^\mu\}$ [32]. Consequently, we can always use analytic signals, instead of real-valued EM fields, to describe wave behaviors. It is worth noting that the value of an analytic signal at each single time point cannot be locally determined by the corresponding instantaneous real-valued EM field, but involves the information of the whole time sequence. As we will show later in the paper, the Noether current of analytic signals associated with translation symmetry gives an E-M tensor distinct from that derived with real fields, which coincides with the time-averaged E-M tensor for monochromatic waves and is useful to describe the evolution of the field envelopes for quasi-monochromatic waves.

The constitutive relation for analytic signals is the same as that of original real fields:

$$\tilde{G}^{\mu\nu}(x^\mu) = \int_0^\infty dk_0 e^{-ik_0x^0} \frac{1}{2} C^{\mu\nu\alpha\beta}(x^i, k_0)(F_\omega)_{\alpha\beta}(x^\mu)$$
$$= \frac{1}{2} C^{\mu\nu\alpha\beta}(x^i, i\partial_0)\tilde{F}_{\alpha\beta}(x^\mu). \quad (20)$$

In many practical scenarios, a complex-valued bianisotropic material is regarded as nondispersive, i.e., treating the constitutive tensor $C^{\mu\nu\alpha\beta}(x^i)$ $(\omega > 0)$ as frequency independent, in a certain positive frequency range of interests. In these cases, an important advantage of using analytic signals is that the constitutive relations for such materials always follow the linear form in time domain:

$$\tilde{G}^{\mu\nu}(x^\mu) = \frac{1}{2} C^{\mu\nu\alpha\beta}(x^i)\tilde{F}_{\alpha\beta}(x^\mu). \quad (21)$$

As a result, Maxwell's equations for analytic signals are simply first-order partial differential equations with respect to $\tilde{F}_{\alpha\beta}$, while Maxwell's equations for real-valued EM fields $F_{\alpha\beta}$ are still temporally nonlocal partial differential equations with infinite orders of time derivatives.

3. EM LAGRANGIAN IN DISPERSIVE BIANISOTROPC MEDIA

A. Lagrangian for Real Fields

To derive the conserved quantity with a momentum dimension corresponding to the space translation symmetry, it is necessary to write the Lagrangian density (action) of EM fields in the generic dispersive bianisotropic media in the first place. For nondissipative real-valued materials, the EM Lagrangian has been well established [1,10,11,16]. However, given that the constitutive relations for dispersive bianisotropic media are intrinsically nonlocal in time, we should check whether or to what extent the variation derivative of the conventional expression of the EM Lagrangian can give the correct Maxwell's equations in such general cases.

According to Eq. (11), the EM Lagrangian density for general dispersive bianisotropic media should be regarded as a function involving infinite orders of derivatives of the gauge potential $\partial_0^n \partial_\nu A_\mu$ $(n = 0, \ldots, \infty)$:

$$\mathcal{L}\left(A_\mu, \partial_\nu A_\mu, \partial_0\partial_\nu A_\mu, \partial_0^2\partial_\nu A_\mu, \cdots, x^\mu\right)$$
$$= -\frac{1}{8c} F_{\mu\nu} C^{\mu\nu\alpha\beta}(x^i, i\partial_0) F_{\alpha\beta} + \frac{1}{c} A_\mu J^\mu$$
$$= -\frac{1}{4c} F_{\mu\nu} G^{\mu\nu} + \frac{1}{c} A_\mu J^\mu = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{H}) + (\mathbf{A}\cdot\mathbf{j} - \varphi\rho). \quad (22)$$

And $\mathcal{S}[A] = \int \mathrm{d}^4x \mathcal{L}(A, \nabla A, \ldots, \nabla^n A, \ldots, x^\mu)$ gives the action of EM fields. Therefore, the functional derivative of the action should be calculated considering the derivative terms of all orders [36–38]:

$$\frac{\delta\mathcal{S}}{\delta A_\mu} = \frac{\partial\mathcal{L}}{\partial A_\mu} + \sum_{m=0}^\infty (-1)^{m+1}\partial_0^m\partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_0^m\partial_\nu A_\mu)}$$
$$= \frac{1}{c}J^\mu - \frac{1}{2c}\partial_\nu G^{\nu\mu} - \frac{1}{4c}\partial_\nu \sum_{m=0}^\infty (-i\partial_0)^m F_{\alpha\beta} C_n^{\alpha\beta\nu\mu}$$
$$= \frac{1}{c}J^\mu - \frac{1}{2c}\partial_\nu G^{\nu\mu} - \frac{1}{4c}\partial_\nu \sum_{m=0}^\infty C_n^{\alpha\beta\nu\mu*}(i\partial_0)^m F_{\alpha\beta}$$
$$= \frac{1}{c}J^\mu - \frac{1}{2c}\partial_\nu G^{\nu\mu} - \frac{1}{4c}\partial_\nu C^{\alpha\beta\nu\mu}(x^i, -i\partial_0)^* F_{\alpha\beta}, \quad (23)$$

where in the second to last step, Eq. (16) has been used. When the material satisfies the lossless condition Eq. (15), $C^{\alpha\beta\mu\nu}(x^i, -i\partial_0)^* = C^{\mu\nu\alpha\beta}(x^i, i\partial_0)$. The higher-derivative Euler–Lagrange (E-L) Eq. (23), which corresponds to the extrema of the action, leads to exactly Maxwell's equation

$$0 = \frac{\delta\mathcal{S}}{\delta A_\mu} = \frac{1}{c}J^\mu - \frac{1}{c}\partial_\nu G^{\nu\mu} \Rightarrow \partial_\nu G^{\nu\mu} = J^\mu. \quad (24)$$

Therefore, the EM Lagrangian in Eq. (22) can correctly describe the dynamics of EM fields in general dispersive bianisotropic materials but respecting Eq. (15), which in return confirms that Eq. (15) [and equivalently Eq. (4)] describes the condition of nondissipation. Indeed, it is not possible to write the Lagrangian in dissipative media that do not explicitly depend on time as Eq. (22); otherwise, we would obtain a conserved EM energy from the Lagrangian in lossy media.

In previous studies on the EM Lagrangian in nonconservative isotropic media [39–44], the EM susceptibility of background materials was always modeled as a separated dynamic field, where dissipation was introduced into Lagrangian formalism using either a reservoir field coupled to the system [39–42] or an *ad hoc* Rayleigh potential [43,44]. However, all these approaches rely on the microscopic oscillator models of the lossy media and hence deviate from our macroscopic methodology representing media purely by constitutive tensors. How to incorporate dissipation into Lagrangian formalism at the macroscopic level is itself a highly nontrivial subject, which is beyond the scope of this work.

B. Lagrangian for Analytic Signals

Now, we introduce an effective real-valued Lagrangian density for the analytic signals

$$\begin{aligned}
&\tilde{\mathcal{L}}\left(\tilde{A}_\mu, \tilde{A}^*_\mu, \partial_\nu \tilde{A}_\mu, \partial_\nu \tilde{A}^*_\mu, \cdots, \partial_0^n \partial_\nu \tilde{A}_\mu, \cdots, x^\mu\right) \\
&= -\frac{1}{16c}\tilde{F}^*_{\mu\nu} C^{\mu\nu\alpha\beta}(x^i, i\partial_0)\tilde{F}_{\alpha\beta} + \frac{1}{2c}\tilde{A}^*_\mu \tilde{J}^\mu + \text{c.c.} \\
&= \frac{1}{4}(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} - \tilde{\mathbf{B}}^* \cdot \tilde{\mathbf{H}}) - \frac{1}{2}(\tilde{\varphi}^* \tilde{\rho} + \tilde{\mathbf{A}}^* \cdot \tilde{\mathbf{j}}) + \text{c.c.}, \qquad (25)
\end{aligned}$$

where "c.c." represents the complex conjugate of the explicitly listed terms, and $\tilde{\mathcal{S}}[A, A^*] = \int d^4x \tilde{\mathcal{L}}(A, A^*, \nabla A, \nabla A^*, \ldots, x^\mu)$ gives the action for analytic signals.

For complex-valued fields, $\tilde{A}_\mu$ and $\tilde{A}^*_\mu$ should be treated as two linearly independent variables. The E-L equation with respect to $\tilde{A}^*_\mu$ reproduces the 4D Maxwell's equation of the analytic signals

$$\begin{aligned}
0 &= \frac{\delta \tilde{\mathcal{S}}}{\delta \tilde{A}^*_\mu} = \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{A}^*_\mu} + \sum_{m=0}^{\infty} (-1)^{m+1} \partial_0^m \partial_\nu \frac{\partial \tilde{\mathcal{L}}}{\partial\left(\partial_0^m \partial_\nu \tilde{A}^*_\mu\right)} \\
&= \frac{1}{2c}\tilde{J}^\mu - \frac{1}{4c}\partial_\nu \tilde{G}^{\nu\mu} - \frac{1}{8c}\partial_\nu \sum_{m=0}^{\infty} (i\partial_0)^m \tilde{F}_{\alpha\beta} C_n^{\alpha\beta\nu\mu *} \\
&= \frac{1}{2c}\tilde{J}^\mu - \frac{1}{4c}\partial_\nu \tilde{G}^{\nu\mu} - \frac{1}{8c}\partial_\nu C^{\alpha\beta\mu\nu}\left(x^i, -i\partial_0\right)^* F_{\alpha\beta} \\
&= \frac{1}{2c}\left[\tilde{J}^\mu - \partial_\nu \tilde{G}^{\nu\mu}\right]. \qquad (26)
\end{aligned}$$

The E-L equation with respect to A^*_μ is identical to the complex conjugate of Eq. (26). This result confirms the validity of the effective Lagrangian (25).

4. ENERGY–MOMENTUM CONSERVATION DERIVED FROM NOETHER'S THEOREM

In studies of the E-M balance for EM waves traveling in a medium, there are two distinct approaches [11]. In the first approach, the EM fields and the medium constitute an isolated system; hence, the total energy and momentum of the system must be conserved, and this is a suitable model to analyze the interaction between EM fields and the medium. In the second approach, the medium is regarded as a non-dynamic background given by the predetermined function $C^{\mu\nu\alpha\beta}(\mathbf{r}, \omega)$. In this situation, it is convenient to analyze the conservation of Minkowski-type quantities in relation to the symmetry of the background medium. Here, we are particularly interested in the second one. In this part, we will follow the standard procedure [11,45,46] to derive the E-M balance equation associated with the Minkowski-type E-M tensor for EM fields in dispersive biansitropic media for both real fields and complex analytic signals.

A. Minkowski Energy–Momentum Tensor for Real Fields

According to Noether's theorem, if the action of the system $\mathcal{S}[A] = \int \mathrm{d}^4x \mathcal{L}(A, \nabla A, \cdots, x)$ is invariant under an infinitesimal symmetry transformation, there exists a conservation law associated with this symmetry transformation. In the presence of spacetime translation symmetry, the corresponding conserved quantities are energy and momentum. In our case, the action is not invariant under spacetime translation, unless the medium $C^{\mu\nu\alpha\beta}$ is homogeneous in spacetime and the four-current J^μ is a constant. Nevertheless, even for inhomogeneous medium $C^{\mu\nu\alpha\beta}(x^i, k_0)$, we can still obtain the E-M balance equation by treating the 4D electric current J and the coefficients of constitutive tensor Eq. (12), $C_n^{\mu\nu\alpha\beta}$ $(n = 1, \ldots, \infty)$ as external non-dynamic fields [11], namely,

$$\mathcal{S}[A] = \mathcal{S}_e[A, J, C_n], \qquad (27)$$

$$\begin{aligned}
&\mathcal{L}\left(A_\mu, \partial_\nu A_\mu, \ldots, x^\mu\right) \\
&= \mathcal{L}_e\left(A_\mu, \partial_\nu A_\mu, \ldots, J^\mu, C_0^{\mu\nu\alpha\beta}, \ldots, C_n^{\mu\nu\alpha\beta}, \ldots\right). \qquad (28)
\end{aligned}$$

Since the new defined function $\mathcal{L}_e$ does not depend explicitly on spacetime coordinates, the E-M balance equation in generic spatially inhomogeneous media can be derived from the variation of $\mathcal{L}_e$ with respect to the infinitesimal spacetime translation $x^\mu \to x^\mu + \epsilon^\mu$. For a Lagrangian with higher-order derivatives $\mathcal{L}(V_i, \nabla V_i, \ldots, \nabla^n V_i, \ldots)$, the general form of the functional derivative with respect to ϵ^μ is given by

$$\begin{aligned}
\frac{\delta \mathcal{S}}{\delta \epsilon^\nu} = -\frac{\delta \mathcal{S}}{\delta V_i}\partial_\nu V_i + \partial_\mu \Bigg[\mathcal{L}\delta^\mu_\nu \\
+ \sum_{m=1}^{\infty}\sum_{n=1}^{m} (-1)^n \delta^\mu_{\alpha_n}\left(\partial_{\alpha_1\cdots\alpha_{n-1}} \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha_1\cdots\alpha_m} V_i)}\right)\partial_{\alpha_{n+1}\cdots\alpha_m}\partial_\nu V_i\Bigg]. \qquad (29)
\end{aligned}$$

Because $\mathcal{L}_e$ does not explicitly depend on x^μ, variation of $\mathcal{S}_e$ with respect to the infinitesimal translation vanishes, and hence $\delta\mathcal{S}_e/\delta\epsilon_\nu = 0$ gives the E-M balance equation. For the EM Lagrangian, Eq. (29) gives

$$\partial_\mu\left(-\mathcal{L}_e\delta^\mu_\nu+\frac{\partial\mathcal{L}_e}{\partial(\partial_\mu A_\sigma)}\partial_\nu A_\sigma+\sum_{m=1}^{\infty}(-1)^m\left(\partial_0^m\frac{\partial\mathcal{L}_e}{\partial(\partial_\mu A_\sigma)}\right)\partial_\nu A_\sigma\right.$$
$$\left.-\delta^\mu_0\sum_{m=1}^{\infty}\sum_{n=1}^{m}(-1)^n\left(\partial_0^{n-1}\frac{\partial\mathcal{L}_e}{\partial(\partial_0^m\partial_\tau A_\sigma)}\right)\partial_0^{m-n}\partial_\tau\partial_\nu A_\sigma\right]$$
$$=-\frac{\delta\mathcal{S}_e}{\delta A_\mu}\partial_\nu A_\mu-\sum_{n=0}^{\infty}\frac{\delta\mathcal{S}_e}{\delta C_n^{\sigma\tau\alpha\beta}}\partial_\nu C_n^{\sigma\tau\alpha\beta}-\frac{\delta\mathcal{S}_e}{\delta J^\mu}\partial_\nu J^\mu. \tag{30}$$

As $\mathcal{A}_\mu$ is a dynamic variable satisfying the E-L equations (24), its variational derivatives vanish $\delta\mathcal{S}_e/\delta A_\mu=\delta\mathcal{S}/\delta A_\mu=0$, whereas $C_n^{\mu\nu\alpha\beta}$, J^μ are non-dynamical variables, so their variational derivatives are, in general, nonzero. Substituting Eq. (25) into Eq. (30), we obtain

$$\partial_\mu\left[\frac{1}{4c}F_{\alpha\beta}G^{\alpha\beta}\delta^\mu_\nu-\frac{1}{c}A_\alpha J^\alpha\delta^\mu_\nu+\frac{1}{c}G^{\mu\alpha}\partial_\nu A_\alpha\right.$$
$$\left.+\delta^\mu_0\frac{-i}{8c}\sum_{m=1}^{\infty}\sum_{n=1}^{m}\left((-i\partial_0)^{n-1}F_{\alpha\beta}\right)C_m^{\alpha\beta\sigma\tau}\left((i\partial_0)^{m-n}\partial_\nu F_{\sigma\tau}\right)\right]$$
$$=\frac{1}{8c}F_{\mu\sigma}\left(\partial_\nu C^{\mu\sigma\alpha\beta}\right)F_{\alpha\beta}-\frac{1}{c}A_\mu\partial_\nu J^\mu. \tag{31}$$

After simplification, Eq. (31) is rewritten as

$$\partial_\mu\Theta^\mu{}_\nu=\frac{1}{8c}F_{\mu\sigma}\left(\partial_\nu C^{\mu\sigma\alpha\beta}\right)F_{\alpha\beta}+\frac{1}{c}J^\mu\partial_\nu A_\mu. \tag{32}$$

Here,

$$\Theta^\mu{}_\nu=\frac{1}{4c}F_{\alpha\beta}G^{\alpha\beta}\delta^\mu_\nu+\frac{1}{c}G^{\mu\alpha}\partial_\nu A_\alpha$$
$$+\delta^\mu_0\frac{1}{8c}\sum_{m=1}^{\infty}\sum_{n=1}^{m}(-1)^n\left(\partial_0^{n-1}F_{\alpha\beta}\right)i^m C_m^{\alpha\beta\sigma\tau}\left(\partial_0^{m-n}\partial_\nu F_{\sigma\tau}\right) \tag{33}$$

gives the canonical E-M tensor (Noether current associated with spacetime translation symmetry) of EM fields in general dispersive bianisotropic media, where the first line, denoted as Θ_0, coincides with the result in nondispersive media [11], while the second line arises entirely from the frequency dispersion of the materials. On account of $C_m^{\alpha\beta\sigma\tau *}=(-1)^m C_m^{\alpha\beta\sigma\tau}$, the obtained E-M tensor is purely real.

According to Eq. (33), the value of the canonical E-M tensor is gauge dependent, i.e., it depends explicitly on the gauge potential. In fact, the definition of the local E-M tensor permits the freedom of adding a total differential term:

$$T^\mu{}_\nu=\Theta^\mu{}_\nu+\partial_\sigma\xi^{\sigma\mu}{}_\nu, \tag{34}$$

provided that $\xi^{\sigma\mu}{}_\nu=-\xi^{\mu\sigma}{}_\nu$. Since $\partial_\mu\partial_\sigma\xi^{\sigma\mu}{}_\nu\equiv 0$, $T^\mu{}_\nu$ respects the same E-M balance equation (32) as $\Theta^\mu{}_\nu$. For general spin-carrying fields, $\xi^{\sigma\mu}{}_\nu$ can be properly selected as

$$\xi^{\sigma\mu}_\nu=\frac{1}{2}\left(S^{\sigma\mu}_\nu+S_\nu{}^{\sigma\mu}-S^{\mu\sigma}_\nu\right), \tag{35}$$

in terms of the corresponding 4D spin angular momentum current $S^{\sigma\mu}_\nu=M^{\sigma\mu}_\nu-(\Theta^\sigma_\nu x^\mu-\Theta^\sigma{}\mu x_\nu)$, with $M^{\sigma\mu}_\nu$ denoting the canonical total angular momentum and $(\Theta^\sigma_\nu x^\mu-\Theta^\sigma{}\mu x_\nu)$ denoting the orbital angular momentum. Then the obtained E-M tensor T^μ_ν, dubbed the Belinfante-Rosenfeld (BR) tensor, is gauge invariant [11,45,46]. In our case, because the gauge-dependent terms appear only in $\Theta_0{}^\mu_\nu$, the elimination of gauge dependence can be already achieved by using the spin angular momentum of the dispersionless case [11,46], i.e., we may select

$$S^{\sigma\mu}_\nu=\frac{1}{c}\left(G^{\sigma\mu}A_\nu-G^\sigma_\nu A^\mu\right), \tag{36}$$

$$\Rightarrow\xi^{\sigma\mu}{}_\nu=\frac{1}{c}G^{\sigma\mu}A_\nu, \tag{37}$$

which obviously satisfies $\xi^{\sigma\mu}_\nu=-\xi^{\mu\sigma}_\nu$. After "relocalization" of the E-M tensor with Eq. (34), the modified BR E-M tensor takes the following form:

$$T^\mu_\nu=\underbrace{\frac{1}{4c}F_{\alpha\beta}G^{\alpha\beta}\delta^\mu_\nu+\frac{1}{c}G^{\mu\alpha}F_{\alpha\nu}+\delta^\mu_0\frac{1}{8c}\sum_{m=1}^{\infty}\sum_{n=1}^{m}(-1)^n\left(\partial_0^{n-1}F_{\alpha\beta}\right)i^m C_m^{\alpha\beta\sigma\tau}\left(\partial_0^{m-n}\partial_\nu F_{\sigma\tau}\right)}_{\text{Minkowski-type E-M tensor: } T_{\text{Min}}{}^\mu_\nu}$$
$$+\frac{1}{c}A_\nu J^\mu. \tag{38}$$

The first part of the BR tensor enclosed by the under-brace depends only on the gauge field $F^{\mu\nu}$, and hence it is gauge invariant, which is indeed the generalized Minkowski E-M tensor of EM fields in dispersive bianisotropic materials. The second part contributed by the four-current J^μ will be separated from the E-M tensor later. Replacing the 4D EM fields with their 3D counterparts, the generalized Minkowski E-M tensor of EM fields can be written as

$$(T_{\text{Min}}{}^\mu{}_\nu)=\left(\begin{array}{c|c} W & -c\mathbf{p}_{\text{Min}}^\intercal \\ \hline \frac{1}{c}\mathbf{S} & \overleftrightarrow{\sigma}_{\text{Min}}\end{array}\right), \tag{39}$$

where the four blocks are, respectively,

(1) energy density:

$$W=\frac{1}{2}\left(\mathbf{D}\cdot\mathbf{E}+\mathbf{H}\cdot\mathbf{B}\right)$$
$$+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}(-1)^n i^m\left(\partial_t^n\Psi^\intercal\right)\hat{\sigma}_3\overleftrightarrow{N}_m\left(\partial_t^{m-n}\Psi\right), \tag{40}$$

with $\Psi=(\mathbf{E},\mathbf{B})^\intercal$ and $\overleftrightarrow{N}_m$ being the series coefficients of the constitutive tensor $\overleftrightarrow{N}(\mathbf{r},\omega)$ in Eq. (7), and $\hat{\sigma}_3=\text{diag}(1,-1)$ being the third Pauli matrix;

(2) energy flux density (Poynting vector):

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}; \tag{41}$$

(3) Minkowski momentum density:

$$\mathbf{p}_{\rm Min} = \mathbf{D}\times\mathbf{B} + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{m}(-1)^n i^m \left(\partial_t^{n-1}\Psi^{\intercal}\right)\hat{\sigma}_3\overleftrightarrow{N}_m\left(\partial_t^{m-n}\partial_j\Psi\right)\mathbf{e}^j, \tag{42}$$

with $\mathbf{e}^j$ denoting the spatial coordinate unit vectors;

(4) Minkowski EM stress tensor:

$$\overleftrightarrow{\sigma}_{\rm Min} = \mathbf{D}\otimes\mathbf{E} + \mathbf{B}\otimes\mathbf{H} - \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{B})\overleftrightarrow{I}. \tag{43}$$

Comparing Eqs. (40)–(43) with their counterparts in nondispersive media [11,16], we find that the frequency dispersion of the materials modifies the expressions of energy density and Minkowski momentum density by adding the additional series in Eq. (40) and Eq. (42), while the expressions of the energy flux density and the Minkowski stress tensor are identical to the nondispersive case.

In terms of the Minkowski-type tensor and the charge conservation $\partial_\mu J^\mu = 0$, we rewrite the E-M balance equation (32) as

$$\partial_\mu T_{\rm Min}{}^{\mu}_{\nu} = \underbrace{\frac{1}{8c}F_{\mu\sigma}\left(\partial_\nu C^{\mu\sigma\alpha\beta}\right)F_{\alpha\beta}}_{f_{(\rm M)\nu}} + \underbrace{\frac{1}{c}J^\mu F_{\mu\nu}}_{f_{(\rm L)\nu}}, \tag{44}$$

where $f_{(\rm M)\nu}$ indicates an effective 4D force density arising from the inhomogeneity of the medium, and $f_{(\rm L)\nu}$ corresponds to 4D Lorentz force density. According to Eq. (44), $\partial_\mu T_{\rm Min}{}^{\mu}_{0} = f_{(\rm L)0}$ ($f_{(\rm M)0} = 0$ since the medium is supposed to be homogeneous in time) gives the energy equilibrium relation

$$\frac{\partial}{\partial t}W + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{j}. \tag{45}$$

And $\partial_\mu T_{\rm Min}{}^{\mu}_{i} = f_{(\rm M)i} + f_{(\rm L)i}$ gives the Minkowski-type momentum equilibrium equation

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{p}_{\rm Min} - \nabla\cdot\overleftrightarrow{\sigma}_{\rm Min} &= -(\rho\mathbf{E} + \mathbf{j}\times\mathbf{B}) - \frac{1}{2}\Psi^{\intercal}\left(\partial_i\overleftrightarrow{N}\right)\Psi\hat{\mathbf{e}}^i \\ &= -(\rho\mathbf{E} + \mathbf{j}\times\mathbf{B}) - \Big[\frac{1}{2}E_i(\nabla\tilde{\varepsilon}^{ij})E_j \\ &\quad - \frac{1}{2}B^i(\nabla\mu_{ij}^{-1})B^j + E_i\mathrm{Re}(\nabla\tilde{\chi}_j^i)B^j\Big].\end{aligned} \tag{46}$$

Equations (45) and (46) can be directly verified by substituting the 3D Maxwell's equations and Eqs. (40)–(43) into the equations.

In source-free and homogeneous media, $f_{(\rm M)} = f_{(\rm L)} = 0$; consequently, the Noether's theorem ensures that the Minkowski-type E-M current is conserved:

$$\partial_\mu T_{\rm Min}{}^{\mu}_{\nu} = 0 \Leftrightarrow \begin{cases}\dfrac{\partial}{\partial t}W + \nabla\cdot\mathbf{S} = 0, \\ \dfrac{\partial}{\partial t}\mathbf{p}_{\rm Min} - \nabla\cdot\overleftrightarrow{\sigma}_{\rm Min} = 0.\end{cases} \tag{47}$$

Integrating Eq. (47) over the whole 3D space, we obtain

$$\frac{d}{dt}W_{\rm total} = \int_V d^3x\, W = -\int_{\partial V} d\mathbf{s}\cdot\mathbf{S} = 0, \tag{48}$$

$$\frac{d}{dt}\mathbf{P}_{\rm Min-total} = \int_V d^3x\,\mathbf{p}_{\rm Min} = \int_{\partial V} d\mathbf{s}\cdot\overleftrightarrow{\sigma}_{\rm Min} = 0, \tag{49}$$

where $d\mathbf{s}$ denotes the surface element on the infinite boundary ∂V; and $\mathbf{S}$, $\overleftrightarrow{\sigma}_{\rm Min}$ are assumed to decrease fast enough at infinity so that their surface integrals over ∂V vanish. Therefore, we have proved that the total energy and the total Minkowski-type momentum corresponding to the instantaneous EM fields are invariant with time in source-free and spatiotemporally homogeneous media:

$$W_{\rm total} = \text{const.}, \qquad \mathbf{P}_{\rm Min-total} = \text{const.} \tag{50}$$

B. Time-Averaged Expressions for Monochromatic Waves

As far as we know, these generalized expressions for generic dispersive bianisotropic media have never appeared in literature. Nevertheless, the correctness of the expressions can be verified via comparing with some special cases given in literature. First, for dispersive isotropic media, i.e., $\overleftrightarrow{N}(\mathbf{r},\omega) = \mathrm{diag}(\varepsilon(\mathbf{r},\omega), \mu^{-1}(\mathbf{r},\omega))$, we can check that Eq. (40) and Eq. (42) can be precisely reduced to the corresponding formulas in Ref. [10]. Second, for monochromatic waves $\tilde{\boldsymbol{E}} = \mathbf{E}_0(\mathbf{r})\exp(-i\omega t)$ (note that it is indeed the analytic signal of the monochromatic wave), we can obtain the time-averaged energy density from Eq. (40) by substituting $i\partial_t \to \omega$ and $f(t)g(t) \to \frac{1}{2}\tilde{f}^*\tilde{g}$ [here, $f(t)$, $g(t)$ denote two time-harmonic functions at frequency ω]:

$$\begin{aligned}\langle W\rangle &= \frac{1}{4}\left(\tilde{\mathbf{D}}^*\cdot\tilde{\mathbf{E}} + \tilde{\mathbf{H}}\cdot\tilde{\mathbf{B}}^*\right) + \frac{1}{4}\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\tilde{\Psi}^\dagger\hat{\sigma}_3\overleftrightarrow{N}_m\omega^m\tilde{\Psi} \\ &= \frac{1}{4}\tilde{\Phi}^\dagger\overleftrightarrow{M}(\mathbf{r},\omega)\tilde{\Phi} + \frac{\omega}{4}\tilde{\Psi}^\dagger\hat{\sigma}_3\frac{\partial\overleftrightarrow{N}(\mathbf{r},\omega)}{\partial\omega}\tilde{\Psi} \\ &= \frac{1}{4}\tilde{\Phi}^\dagger\frac{\partial\left(\omega\overleftrightarrow{M}(\mathbf{r},\omega)\right)}{\partial\omega}\tilde{\Phi},\end{aligned} \tag{51}$$

where $\tilde{\Phi} = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})^{\intercal} = \overleftrightarrow{R}^{-1}\tilde{\Psi}$, with $\overleftrightarrow{R} = \begin{pmatrix}\overleftrightarrow{I} & 0 \\ \overleftrightarrow{\chi}^\dagger & \overleftrightarrow{\mu}\end{pmatrix}$, and the identity $\overleftrightarrow{R}^\dagger\hat{\sigma}_3(\partial_\omega\overleftrightarrow{N})\overleftrightarrow{R} = \partial_\omega\overleftrightarrow{M}$ has been used in the last step. The obtained expression is exactly the generalized Brillouin's formula for time-averaged EM energy density in dispersive media [47], thereby verifying our results.

In vacuum, the relocalization procedure can guarantee the obtained BR tensor is symmetric; however, in media, the BR tensor as well as the Minkowski-type tensor are asymmetric in general. Consequently, the direction of the Minkowski-type momentum $\mathbf{p}_{\rm Min}$ does not coincide with the energy flux $\mathbf{S}$ in generic bianisotropic media. Similarly, from Eq. (42), we can obtain the time-averaged Minkowski momentum density for monochromatic waves:

$$\langle \mathbf{p}_{\text{Min}} \rangle = \frac{1}{2}\text{Re}(\tilde{\mathbf{D}}^* \times \tilde{\mathbf{B}}) + \frac{1}{4}\text{Re}\sum_{m=1}^{\infty}\sum_{n=1}^{m}(-1)^n i^m \left(\partial_t^{n-1}\tilde{\Psi}^{\intercal}\right)^* \hat{\sigma}_3 \overleftrightarrow{N}_m \left(\partial_t^{m-n}\partial_j\tilde{\Psi}\right)\mathbf{e}^j$$
$$= \frac{1}{2}\text{Re}(\tilde{\mathbf{D}}^* \times \tilde{\mathbf{B}}\rangle + \frac{1}{4}\text{Re}\sum_{m=1}^{\infty}\sum_{n=1}^{m}(-i)\tilde{\Psi}^\dagger\hat{\sigma}_3\overleftrightarrow{N}_m\omega^{m-1}\partial_j\tilde{\Psi}\mathbf{e}^j$$
$$= \frac{1}{2}\text{Re}(\tilde{\mathbf{D}}^* \times \tilde{\mathbf{B}}\rangle + \frac{1}{4}\text{Im}\left(\tilde{\Psi}^\dagger\hat{\sigma}_3\frac{\partial\overleftrightarrow{N}(\mathbf{r},\omega)}{\partial\omega}\partial_j\tilde{\Psi}\right)\mathbf{e}^j. \tag{52}$$

This formula generalizes the previous result in isotropic media [10,13] to the extended cases of bianisotropic media. If the field is a plane wave $\tilde{\mathbf{E}} \sim \exp(i(\mathbf{k}\cdot\mathbf{r}-\omega t))$ in a homogeneous medium, Eq. (52) is further reduced to (see Appendix A for derivation)

$$\langle \mathbf{p}_{\text{Min}} \rangle = \frac{1}{4\omega}\left(\tilde{\Phi}^\dagger\frac{\partial(\omega\overleftrightarrow{M}(\omega))}{\partial\omega}\tilde{\Phi}\right)\mathbf{k} = \frac{\langle W\rangle}{\omega}\mathbf{k}, \tag{53}$$

which indicates that the time-averaged Minkowski momentum of a plane wave is always parallel to the direction of the wave vector $\mathbf{k}$. In contrast, the direction of the energy flux [Eq. (41)] can deviate from the wave vector in anisotropic media. Moreover, Eq. (53) clearly demonstrates that the correspondence between the macroscopic Minkowski momentum density and the canonical momentum of a single photon $\hbar\mathbf{k}$, which was previously derived in isotropic media [5,12], remains valid in general dispersive bianisotropic materials:

$$\langle \mathbf{p}_{\text{Min}} \rangle = N_{\text{photon}}\hbar\mathbf{k}, \tag{54}$$

where $N_{\text{photon}} = \langle W\rangle/(\hbar\omega)$ denotes the number density of photons in the material. On the other hand, we can also prove that the following relation between the group velocity $\mathbf{v}_g$ and the energy flux remains established for general homogeneous dispersive bianisotropic materials (see Appendix B for derivation):

$$\mathbf{v}_g = \nabla_{\mathbf{k}}\omega = \frac{\langle\mathbf{S}\rangle}{\langle W\rangle}. \tag{55}$$

We note that a similar expression is also valid for surface waves supported on the surfaces of a dispersive bianisotropic material [12,13,48,49].

1. *Remarks on Spin-Orbital Decomposition*

In vacuum [50–53] and more generally in source-free isotropic dispersive media with $\overleftrightarrow{M} = \text{diag}(\varepsilon,\mu)$ [12,13], the time-averaged Minkowski momentum can be elegantly reexpressed using spin–orbital decomposition:

$$\langle \mathbf{p}_{\text{Min}} \rangle \stackrel{\text{isotropic}}{=\!=\!=} \langle \mathbf{p}_{\text{orb}} \rangle + \langle \mathbf{p}_{\text{spin}} \rangle. \tag{56}$$

Here,

$$\langle \mathbf{p}_{\text{orb}} \rangle = \frac{1}{4\omega}\text{Im}\left(\frac{\partial(\omega\varepsilon)}{\partial\omega}\tilde{\mathbf{E}}^*\cdot(\nabla)\tilde{\mathbf{E}} + \frac{\partial(\omega\mu)}{\partial\omega}\tilde{\mathbf{H}}^*\cdot(\nabla)\tilde{\mathbf{H}}\right)$$
$$= \frac{1}{4\hbar\omega}\text{Re}\left(\tilde{\Phi}^\dagger\frac{\partial\left(\omega\overleftrightarrow{M}(\omega)\right)}{\partial\omega}(-i\hbar\nabla)\tilde{\Phi}\right) \tag{57}$$

denotes the orbital momentum density that generalizes the expression (53) for a plane wave by replacing the wave vector with the quantum mechanic canonical momentum operator $\hat{\boldsymbol{p}}_{\text{can}} = -i\hbar\nabla$ for any inhomogeneous fields. The spin part,

$$\langle \mathbf{p}_{\text{spin}} \rangle = \frac{1}{2\omega}\nabla\times\langle \mathsf{S}_{\text{Min}}\rangle, \tag{58}$$

is associated with a "naïve" Minkowski spin angular momentum density [13]:

$$\langle \mathsf{S}_{\text{Min}}\rangle = \frac{1}{2}\text{Im}\left(\varepsilon\tilde{\mathbf{E}}^*\times\tilde{\mathbf{E}} + \mu\tilde{\mathbf{H}}^*\times\tilde{\mathbf{H}}\right)$$
$$= \frac{1}{2}\text{Re}\left[\tilde{\Phi}^\dagger\overleftrightarrow{M}\begin{pmatrix}\hat{\mathsf{S}} & 0\\ 0 & \hat{\mathsf{S}}\end{pmatrix}\tilde{\Phi}\right], \tag{59}$$

where $\hat{\mathsf{S}} = \mathsf{S}_l\mathbf{e}^l$ denotes the spin-1 operator with $(\mathsf{S}_l)_{jk} = i\epsilon_{jlk}$. The local spin momentum density $\langle\mathbf{p}_{\text{spin}}\rangle$ vanishes for plane waves and does not contribute to the total time-averaged momentum for localized fields

$$\langle \mathbf{P}_{\text{Min-total}} \rangle \stackrel{\text{isotropic}}{=\!=\!=} \langle \mathbf{P}_{\text{orb-total}} \rangle, \tag{60}$$

as long as the fields decay fast enough at infinity such that $\int_V \mathbf{p}_{\text{spin}}\text{d}V = -\frac{1}{2}\lim_{r\to\infty}\int_{\partial V}\mathsf{S}_{\text{Min}}\times d\mathbf{s} = 0$. In this way, the total orbital momentum is formalized as the expectation value of the momentum operator $\hat{\boldsymbol{p}}_{\text{can}}$ in quantum mechanics, $\langle\mathbf{P}_{\text{orb-total}}\rangle = \frac{1}{4\hbar\omega}\langle\langle\tilde{\Phi}|\hat{\boldsymbol{p}}_{\text{can}}|\tilde{\Phi}\rangle\rangle$, using the modified inner product $\langle\langle\psi|\phi\rangle\rangle = \text{Re}\int_V \text{d}V\psi^\dagger\partial_\omega(\omega\overleftrightarrow{M})\phi$ [54,55].

However, in general bianisotropic materials, if we maintain the definitions of the orbital and spin momentum densities given in the second lines of Eq. (57) and Eq. (59), respectively, two additional terms appear in the spin–orbital decomposition of the Minkowski momentum density [Eq. (52)]:

$$\langle \mathbf{p}_{\text{Min}} \rangle = \langle \mathbf{p}_{\text{orb}} \rangle + \langle \mathbf{p}_{\text{spin}} \rangle + \frac{1}{4}\text{Im}\left(\tilde{\Phi}^\dagger\frac{\partial\overleftrightarrow{M}}{\partial\omega}(\overleftrightarrow{R}^{-1}\nabla\overleftrightarrow{R})\tilde{\Phi}\right) + \frac{1}{4\omega}\nabla\cdot\text{Im}\left(\tilde{\mathbf{D}}^*\otimes\tilde{\mathbf{E}} + \tilde{\mathbf{B}}^*\otimes\tilde{\mathbf{H}} - \tilde{\mathbf{E}}\otimes\tilde{\mathbf{D}}^* - \tilde{\mathbf{H}}\otimes\tilde{\mathbf{B}}^*\right). \tag{61}$$

The appearance of the first additional term [second line in Eq. (61)] associated with the material's inhomogeneity is because

$$\frac{\partial\overleftrightarrow{M}}{\partial\omega}(\overleftrightarrow{R}^{-1}\nabla\overleftrightarrow{R}) = \begin{pmatrix}(\partial_\omega\overleftrightarrow{\chi})\overleftrightarrow{\mu}^{-1}(\nabla\overleftrightarrow{\chi}) & (\partial_\omega\overleftrightarrow{\chi})\mu^{-1}(\nabla\overleftrightarrow{\mu})\\ (\partial_\omega\overleftrightarrow{\mu})\overleftrightarrow{\mu}^{-1}(\nabla\overleftrightarrow{\chi}) & (\partial_\omega\overleftrightarrow{\mu})\overleftrightarrow{\mu}^{-1}(\nabla\overleftrightarrow{\mu})\end{pmatrix}$$

becomes a non-Hermitian matrix in bianisotropic media, but this term vanishes in homogeneous media, and the asymmetricity of the Minkowski stress tensor in anisotrpic media gives rise to the second additional term [third line in Eq. (61)]. Nevertheless, this term has no contribution to the total Minkowski momentum $\langle \mathbf{P}_{\mathrm{Min-total}} \rangle$ for any localized fields.

The intricate decomposition in Eq. (61) for the Minkowski momentum in bianisotrpic materials has less clear physical meaning than Eq. (56) for isotropic media. Whether there exist other proper generalizations of the spin–orbital decomposition for general bianisotropic media invites further studies.

C. Minkowski Energy–Momentum Tensor for Analytic Signals

Following a similar procedure, we can derive the E-M conservation law for complex-valued analytic signals, which is in general distinct from that for real fields.

Regarding the 4D current $\tilde{J}^\mu$, $\tilde{J}^{\mu *}$ and the series coefficients of the constitutive tensor $C_n^{\mu\nu\alpha\beta}$ as independent arguments of the action and Lagrangian for analytic signal Eq. (25) and using Noether's theorem with Eq. (29), we attain the E-M balance equation for analytic signals:

$$\partial_\mu \tilde{\Theta}^\mu_\nu = \frac{1}{8c}\tilde{F}^*_{\mu\sigma}\left(\partial_\nu C^{\mu\sigma\alpha\beta}\right)\tilde{F}_{\alpha\beta} + \frac{1}{2c}\left(\tilde{J}^\mu \partial_\nu \tilde{A}^*_\mu + \tilde{J}^{\mu *}\partial_\nu \tilde{A}_\mu\right), \tag{62}$$

where the canonical 4D E-M tensor for analytic signals is given by

$$\begin{aligned}\tilde{\Theta}^\mu_\nu = {}& \frac{1}{4c}\tilde{F}^*_{\alpha\beta}\tilde{G}^{\alpha\beta}\delta^\mu_\nu + \frac{1}{2c}\left(\tilde{G}^{\mu\alpha *}\partial_\nu \tilde{A}_\alpha + \tilde{G}^{\mu\alpha}\partial_\nu \tilde{A}^*_\alpha\right)\\ &+ \delta^\mu_0 \frac{1}{8c}\sum_{m=1}^{\infty}\sum_{n=1}^{m}\mathrm{Im}\left[\left((i\partial_0)^{n-1}\tilde{F}_{\alpha\beta}\right)^*\right.\\ &\left.\times\, C_m^{\alpha\beta\sigma\tau}\left((i\partial_0)^{m-n}\partial_\nu \tilde{F}_{\sigma\tau}\right)\right].\end{aligned} \tag{63}$$

Adding a total differential term to the E-M tensor $\tilde{T}^\mu_\nu = \tilde{\Theta}^\mu_\nu + \partial_\sigma \tilde{\xi}^{\sigma\mu}_\nu$ with $\tilde{\xi}^{\sigma\mu}_\nu = \frac{1}{2c}(\tilde{G}^{\sigma\mu *}\tilde{A}_\nu + \tilde{G}^{\sigma\mu}\tilde{A}^*_\nu)$, we obtain the BR E-M tensor for analytic signal $\tilde{T}^\mu_\nu = \tilde{T}_{\mathrm{Min}}{}^\mu_\nu + \frac{1}{2c}(\tilde{A}^*_\nu \tilde{J}^\mu + \tilde{A}_\nu \tilde{J}^{\mu *})$, and the first part gives the Minkowski E-M tensor for analytic signals in dispersive bianisotropic materials:

$$\begin{aligned}\tilde{T}_{\mathrm{Min}}{}^\mu{}_\nu = {}& \frac{1}{4c}\tilde{F}^*_{\alpha\beta}\tilde{G}^{\alpha\beta}\delta^\mu_\nu + \frac{1}{2c}\left(\tilde{G}^{\mu\alpha *}\tilde{F}_{\alpha\nu} + \tilde{G}^{\mu\alpha}\tilde{F}^*_{\alpha\nu}\right)\\ &+ \delta^\mu_0 \frac{1}{8c}\sum_{m=1}^{\infty}\sum_{n=1}^{m}\mathrm{Im}\left[\left((i\partial_0)^{n-1}\tilde{F}_{\alpha\beta}\right)^* C_m^{\alpha\beta\tau}\left((i\partial_0)^{m-n}\partial_\nu \tilde{F}_\tau\right)\right]\\ = {}& \begin{pmatrix} \tilde{W} & -c\tilde{\mathbf{p}}^{\mathsf{T}}_{\mathrm{Min}} \\ \frac{1}{c}\tilde{\mathbf{S}} & \overset{\leftrightarrow}{\tilde{\sigma}}_{\mathrm{Min}} \end{pmatrix},\end{aligned} \tag{64}$$

where the four blocks are, respectively, given by

$$\begin{aligned}\tilde{W} = {}& \frac{1}{2}\left(\tilde{\mathbf{D}}^* \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^*\right)\\ &+ \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\left((-i\partial_t)^n \tilde{\Psi}^\dagger\right)\hat{\sigma}_3 \overset{\leftrightarrow}{N}_m \left((i\partial_t)^{m-n}\tilde{\Psi}\right),\end{aligned} \tag{65}$$

$$\tilde{\mathbf{S}} = \mathrm{Re}\left(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}\right), \tag{66}$$

$$\begin{aligned}\tilde{\mathbf{p}}_{\mathrm{Min}} = {}& \mathrm{Re}\left(\tilde{\mathbf{D}}^* \times \tilde{\mathbf{B}}\right) + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{m}\mathrm{Im}\Big[\left((-i\partial_t)^{n-1}\tilde{\Psi}^\dagger\right)\\ &\times \hat{\sigma}_3 \overset{\leftrightarrow}{N}_m \left((i\partial_t)^{m-n}\partial_j \tilde{\Psi}\right)\Big]\mathbf{e}^j,\end{aligned} \tag{67}$$

$$\overset{\leftrightarrow}{\tilde{\sigma}}_{\mathrm{Min}} = \mathrm{Re}\left(\tilde{\mathbf{D}}^* \otimes \tilde{\mathbf{E}} + \tilde{\mathbf{B}}^* \otimes \tilde{\mathbf{H}}\right) + -\frac{1}{2}\left(\tilde{\mathbf{D}}^* \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}\right)\overset{\leftrightarrow}{I}. \tag{68}$$

The 4D E-M balance equation for analytic signals can be expressed with the Minkowski tensor as

$$\partial_\mu \tilde{T}_{\mathrm{Min}}{}^\mu_\nu = \frac{1}{8c}\tilde{F}^*_{\mu\sigma}\left(\partial_\nu C^{\mu\sigma\alpha\beta}\right)\tilde{F}_{\alpha\beta} + \frac{1}{2c}\left(\tilde{J}^{\mu *}\tilde{F}_{\mu\nu} + \tilde{J}^\mu \tilde{F}^*_{\mu\nu}\right). \tag{69}$$

The energy and momentum currents satisfy, respectively,

$$\frac{\partial}{\partial t}\tilde{W} + \nabla\cdot\tilde{\mathbf{S}} = -\mathrm{Re}\left(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{j}}\right), \tag{70}$$

$$\frac{\partial}{\partial t}\tilde{\mathbf{p}}_{\mathrm{Min}} - \nabla\cdot\overset{\leftrightarrow}{\tilde{\sigma}}_{\mathrm{Min}} = -\mathrm{Re}\left(\tilde{\rho}^*\tilde{\mathbf{E}} + \tilde{\mathbf{j}}^* \times \tilde{\mathbf{B}}\right) - \frac{1}{2}\Psi^\dagger\left(\partial_i \overset{\leftrightarrow}{N}\right)\Psi\hat{\boldsymbol{e}}^i. \tag{71}$$

In source-free and homogeneous media, both energy and Minkowski momentum for analytic signals are conserved:

$$\frac{\partial}{\partial t}\tilde{W} + \nabla\cdot\tilde{\mathbf{S}} = 0, \tag{72}$$

$$\frac{\partial}{\partial t}\tilde{\mathbf{p}}_{\mathrm{Min}} - \nabla\cdot\overset{\leftrightarrow}{\tilde{\sigma}}_{\mathrm{Min}} = 0. \tag{73}$$

In what follows, we consider two special cases to clarify the physical meaning of the conserved E-M tensor for analytic signals.

1. Real-Valued Materials

In real-valued materials, i.e., $C^{\alpha\beta\mu\nu}(\mathbf{r}, \omega) \in \mathbb{R}$ at all frequencies, the Minkowski tensor $\tilde{T}_{\mathrm{Min}}{}^\mu_\nu$ for analytic signals can be decomposed into two decoupled parts:

$$\tilde{T}_{\mathrm{Min}}{}^\mu{}_\nu = T_{\mathrm{Min}}{}^\mu{}_\nu + T^{(I)}_{\mathrm{Min}}{}^\mu{}_\nu, \tag{74}$$

where the first part $T_{\mathrm{Min}}{}^\mu_\nu$ is just the Minkowski tensor for instantaneous real field $F_{\mu\nu}$, while the second part purely depends on $F^{(I)}_{\mu\nu}$, i.e., the Hilbert transform of the real field [see Eq. (18)]:

$$T^{(I)\mu}_{\text{Min }\nu} = \frac{1}{4c} F^{(I)}_{\alpha\beta} G^{(I)\alpha\beta} \delta^{\mu}_{\nu} + \frac{1}{c} G^{(I)\mu\alpha} F^{(I)}_{\alpha\nu} + \delta^{\mu}_{0} \frac{1}{8c} \sum_{m=1}^{\infty} \sum_{n=1}^{2m} (-1)^{n+m} \left(\partial_0^{n-1} F^{(I)}_{\alpha\beta}\right) \times C^{\alpha\beta\sigma\tau}_{2m} \left(\partial_0^{2m-n} \partial_\nu F^{(I)}_{\sigma\tau}\right), \tag{75}$$

where only even order series coefficients $C^{\alpha\beta\mu\nu}_{2m}$ exist for a real-valued constitutive tensor. Therefore, the second part is also conserved in a source-free homogeneous real-valued medium: $\partial_\mu T^{(I)\mu}_{\text{Min }\nu} = 0$.

However, in complex-valued bianisotropic materials, $F^{\mu\nu}$ and $F^{(I)}_{\mu\nu}$ are coupled in $\tilde{T}_{\text{Min}}{}^{\mu}_{\nu}$; consequently, the Minkowski tensor $\tilde{T}_{\text{Min}}{}^{\mu}_{\nu}$ for analytic signals is not the simple summation of two conserved E-M tensors in general.

2. Monochromatic and Quasi-Monochromatic Waves

Compared with Eqs. (51) and (52), we observe that, for monochromatic waves $\tilde{\mathbf{E}} \sim \exp(-i\omega t)$, the conserved Minkowski E-M tensor of the analytic signal is precisely twice as big as the time-period-averaged value of the corresponding real field:

$$\tilde{T}_{\text{Min}}{}^{\mu}_{\nu} = 2\langle T_{\text{Min}}{}^{\mu}_{\nu}\rangle. \tag{76}$$

Then Eqs. (69) and (70) reduce to the E-M balance equations for the time-averaged static states.

It is worth emphasizing that the Minkowski-type quantities defined in Eqs. (65)–(68) represent instantaneous quantities associated with arbitrary time-dependent complex-valued EM fields, but not the time-period-averaged values of the real-valued harmonic fields as routinely reported in the literature. In fact, the time-period average is not always well defined for generic time-dependent fields, but for quasi-monochromatic waves, these results for analytic signals are equivalent to the slowly varying envelopes of quasi-period-averaged quantities for real-valued EM fields [32].

5. ABRAHAM MOMENTUM AND CENTROID MOTION

According to the procedure of so-called "Abrahamization" [11,56], the covariant form of the Abraham-type E-M tensor with respect to the real-valued EM fields in an arbitrary inertial reference frame can be written as the following symmetric tensor:

$$\begin{aligned} T_{\text{Abr}}{}^{\mu}_{\nu} &= T_{\text{Min}}{}^{\mu}_{\nu} - \gamma^{\mu\alpha}\frac{1}{2}\left(T_{\text{Min}\,\alpha\nu} - T_{\text{Min}\,\nu\alpha}\right) - \frac{1}{2c^2}\left(T_{\text{Min}}{}^{\mu\alpha} - T_{\text{Min}}{}^{\alpha\mu}\right) u_\alpha u_\nu \\ &= \frac{1}{4c} F_{\alpha\beta} G^{\alpha\beta} \delta^{\mu}_{\nu} + \frac{1}{2c}\left(G^{\mu\alpha} F_{\alpha\nu} + F^{\mu\alpha} G_{\alpha\nu}\right) + \frac{1}{2c^3} u^{\mu} u_{\alpha} \left(G^{\alpha\beta} F_{\beta\nu} - F^{\alpha\beta} G_{\beta\nu}\right) \\ &\quad + \frac{1}{2c^3}\left(F^{\mu\beta} G_{\beta\alpha} - G^{\mu\beta} F_{\beta\alpha}\right) u^{\alpha} u_{\nu} - \frac{1}{2}\left(\delta^{0}_{\nu} t^{\mu} + \delta^{\mu}_{0}\right) + \frac{\gamma}{2c}\left(t^{\mu} u_\nu - u^{\mu} t_\nu\right) + \left(t^{\alpha} u_\alpha\right)\left(u^{\mu}\delta^{0}_{\nu} + u_\nu \delta^{\mu}_{0}\right), \end{aligned} \tag{77}$$

where

$$t_\nu = \frac{1}{8c} \sum_{m=1}^{\infty} \sum_{n=1}^{m} (-1)^n \left(\partial_0^{n-1} F_{\alpha\beta}\right) i^m C^{\alpha\beta\sigma\tau}_{m} \left(\partial_0^{m-n} \partial_\nu F_{\sigma\tau}\right), \tag{78}$$

$(u^\alpha) = \gamma(c, \mathbf{v})$ is the four-velocity of the background medium in the reference frame of interest, $\gamma = (1 - v^2/c^2)^{-1/2}$, and $\gamma^{\mu\alpha} = g^{\mu\alpha} - u^\alpha u^\mu / c^2$ denotes the projection operator onto the local rest frame of the medium.

If the medium is at rest in the laboratory reference, i.e., $\mathbf{v} = 0$ and $u^\alpha = (c, 0, 0, 0)$, the Abraham-type E-M tensor is explicitly expressed as

$$\left(T_{\text{Abr}}{}^{\mu}{}_{\nu}\right) \overset{\text{rest ref.}}{=\!=} \left(\begin{array}{c|c} W & -\frac{1}{c}\mathbf{S} \\ \hline \frac{1}{c}\mathbf{S} & \frac{1}{2}\left(\overleftrightarrow{\sigma}_{\text{Min}} + \overleftrightarrow{\sigma}^{\mathsf{T}}_{\text{Min}}\right) \end{array}\right). \tag{79}$$

As shown, T_{Abr} is a symmetric tensor, which gives the identical energy density and energy flux as those given by the Minkowski-type E-M tensor (39), but the Abraham momentum $\mathbf{p}_{\text{Abr}}$ and stress tensor $\overleftrightarrow{\sigma}_{\text{Abr}}$ are different from the Minkowski-type definitions:

$$\mathbf{p}_{\text{Abr}} = \frac{1}{c^2}\mathbf{S} = \frac{1}{c^2}\left(\mathbf{E} \times \mathbf{H}\right), \tag{80}$$

$$\begin{aligned} \overleftrightarrow{\sigma}_{\text{Abr}} &= \frac{1}{2}\left(\overleftrightarrow{\sigma}_{\text{Min}} + \overleftrightarrow{\sigma}^{\mathsf{T}}_{\text{Min}}\right) \\ &= \frac{1}{2}\left(\mathbf{D} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{D} + \mathbf{B} \otimes \mathbf{H} + \mathbf{H} \otimes \mathbf{B}\right) \\ &\quad - \frac{1}{2}\left(\mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B}\right) \overleftrightarrow{I}. \end{aligned} \tag{81}$$

Thus, the Abraham momentum is proportional to the Poynting vector. Express the momentum equilibrium equation in source-free homogeneous media with Abraham-type quantities:

$$\frac{\partial}{\partial t}\mathbf{p}_{\text{Abr}} - \nabla \cdot \overleftrightarrow{\sigma}_{\text{Abr}} + \mathbf{f}_{\text{Abr}} = 0, \tag{82}$$

where an additional Abraham force appears in the equation

$$\begin{aligned} \mathbf{f}_{\text{Abr}} &= \frac{\partial}{\partial t}\left(\mathbf{D} \times \mathbf{B} - \frac{1}{c^2}\mathbf{E} \times \mathbf{H} - \mathbf{t}\right) \\ &\quad + \frac{1}{2}\nabla \cdot \left(\mathbf{E} \otimes \mathbf{D} - \mathbf{D} \otimes \mathbf{E} + \mathbf{H} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{H}\right) \\ &= \frac{\partial}{\partial t}\left(\mathbf{D} \times \mathbf{B} - \frac{1}{c^2}\mathbf{E} \times \mathbf{H} - \mathbf{t}\right) + \frac{1}{2}\nabla \times \left(\mathbf{D} \times \mathbf{E} + \mathbf{B} \times \mathbf{H}\right), \end{aligned} \tag{83}$$

where $\mathbf{t}$ is the spatial part of the four-vector given in Eq. (78) induced by the material dispersion. Since the first term in the Abraham force cannot be expressed as a divergence for general cases, the total Abraham-type momentum does not conserve even in homogeneous media:

$$\frac{d}{dt}\mathbf{P}_{\mathrm{Abr-total}} = -\int_V \mathrm{d}^3x \mathbf{f}_{\mathrm{Abr}} \neq \mathbf{0}. \tag{84}$$

Being an exception, the total Abraham-type momentum is still conserved in nondispersive homogeneous (bi-)isotropic media, despite the existence of the local Abraham force, because the Abraham momentum is always proportional to the Minkowski momentum $\mathbf{p}_{\mathrm{Abr}} = \mathbf{p}_{\mathrm{Min}}/c^2(\varepsilon\mu - \chi^2)$.

On the other hand, let us consider the instantaneous intensity centroid of a wave packet traveling in source-free space:

$$\mathbf{r}_c = \frac{1}{W_{\mathrm{total}}}\int_V \mathrm{d}^3x\, \mathbf{r} W, \tag{85}$$

with $W_{\mathrm{total}} = \int_V \mathrm{d}^3 W$, and the integration is over the whole space V. The velocity of the centroid satisfies

$$\begin{aligned}\frac{d\mathbf{r}_c}{dt} &= \frac{1}{W_{\mathrm{total}}}\int_V \mathrm{d}^3x\, \mathbf{r}\frac{\partial W}{\partial t} = -\frac{1}{W_{\mathrm{total}}}\int_V \mathrm{d}^3x\, \mathbf{r}\nabla\cdot\mathbf{S} \\ &= \frac{1}{W_{\mathrm{total}}}\int_V \mathrm{d}^3x\, [\mathbf{S} - \nabla\cdot(\mathbf{S}\otimes\mathbf{r})] \\ &= \frac{c^2}{W_{\mathrm{total}}}\mathbf{P}_{\mathrm{Abr-total}} - \underbrace{\frac{1}{W_{\mathrm{total}}}\int_{\partial V} \mathrm{d}\mathbf{s}\cdot\mathbf{S}\otimes\mathbf{r}}_{=0} \\ &= \frac{c^2}{W_{\mathrm{total}}}\mathbf{P}_{\mathrm{Abr-total}}.\end{aligned} \tag{86}$$

Akin to real fields, we can also obtain the Abraham E-M tensor for analytic signals via "Abrahamization." In the rest reference, it is given by

$$(\tilde{T}_{\mathrm{Abr}}{}^{\mu}{}_{\nu}) \underset{\text{rest ref.}}{=\!=} \left(\begin{array}{c|c} \tilde{W} & -\frac{1}{c}\tilde{\mathbf{S}} \\ \hline \frac{1}{c}\tilde{\mathbf{S}} & \frac{1}{2}\left(\overset{\leftrightarrow}{\sigma}_{\mathrm{Min}} + \overset{\leftrightarrow}{\tilde{\sigma}}{}^{\mathsf{T}}_{\mathrm{Min}}\right)\end{array}\right). \tag{87}$$

Hence, the Abraham momentum density and stress tensor for analytic signals read

$$\tilde{\mathbf{p}}_{\mathrm{Abr}} = \frac{1}{c^2}\tilde{\mathbf{S}} = \frac{1}{c^2}\mathrm{Re}\left(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}\right), \tag{88}$$

$$\begin{aligned}\overset{\leftrightarrow}{\tilde{\sigma}}_{\mathrm{Abr}} = \frac{1}{2}\mathrm{Re}\left(\tilde{\mathbf{D}}^*\otimes\tilde{\mathbf{E}} + \tilde{\mathbf{E}}^*\otimes\tilde{\mathbf{D}}^* + \tilde{\mathbf{B}}^*\otimes\tilde{\mathbf{H}} + \tilde{\mathbf{H}}^*\otimes\tilde{\mathbf{B}}\right) \\ - \frac{1}{2}\left(\tilde{\mathbf{D}}^*\cdot\tilde{\mathbf{E}} + \tilde{\mathbf{H}}\cdot\tilde{\mathbf{B}}^*\right)\overset{\leftrightarrow}{I}.\end{aligned} \tag{89}$$

The total Abraham momentum is non-conserved, $d\tilde{\mathbf{P}}_{\mathrm{Abr-total}}/dt \neq 0$, in general bianisotropic media as well. We can further define the centroid of a wave packet of an analytic signal:

$$\begin{aligned}\frac{d\tilde{\mathbf{r}}_c}{dt} &= \frac{1}{\tilde{W}_{\mathrm{total}}}\int_V \mathrm{d}^3x\, \mathbf{r}\frac{\partial\tilde{W}}{\partial t} \\ &= -\frac{1}{\tilde{W}_{\mathrm{total}}}\int_V \mathrm{d}^3x\, \mathbf{r}\nabla\cdot\tilde{\mathbf{S}} \\ &= \frac{c^2}{\tilde{W}_{\mathrm{total}}}\tilde{\mathbf{P}}_{\mathrm{Abr-total}}.\end{aligned} \tag{90}$$

For a quasi-monochromatic wave, the centroid of the analytic signal is identical to the time-averaged centroid in a quasi-period of the wave, i.e., $\tilde{\mathbf{r}}_c = \langle\mathbf{r}_c\rangle$.

Therefore, the velocity of the intensity centroid is proportional to the total Abraham-type momentum (total energy flux), for both instantaneous real EM fields and the time-averaged fields of a quasi-monochromatic wave. Both the speed and moving direction of the centroid can change over time even in homogeneous media. Consequently, the centroid of a wave packet can propagate along a curved trajectory in a generic homogeneous bianisotropic medium. This counterintuitive effect had been predicted recently using phenomenological non-Abelian field theory [24], and what we have derived here gives this effect a rigorous interpretation from the viewpoint of the conservation of the E-M tensor.

6. EXAMPLE—MOMENTUM CONSERVATION IN ZITTERBEWEGUNG EFFECT

In this section, we revisit the Zitterbewegung effect discussed in the Introduction. We note that the Zitterbewegung effect for light may also emerge from the anomalous velocity induced by Berry curvature $\Omega(\mathbf{k})$ (gauge field in the reciprocal space) [57,58]. Nevertheless, the existence of anomalous velocity $d\mathbf{k}/dt \times \Omega(\mathbf{k})$ requires $d\mathbf{k}/dt \neq 0$ and hence the inhomogeneity of the media, which has no contribution to the centroid wavy trajectories of beams in homogeneous bianisotropic media, as shown in Fig. 1(b). Here, we will numerically verify that the generalized Minkowski momentum is conserved during the trembling motion of light, while the Abraham momentum varies with time. We consider a typical homogeneous "non-Abelian" metamaterial [24], whose relative permittivity and permeability tensors are given by

$$\overset{\leftrightarrow}{\varepsilon}(\omega > 0) = \varepsilon_0 \left(\begin{array}{cc|c} \varepsilon_T & 0 & -ia \\ 0 & \varepsilon_T & b \\ \hline ia & b & \varepsilon_z\end{array}\right), \tag{91}$$

$$\overset{\leftrightarrow}{\mu}(\omega > 0) = \mu_0 \left(\begin{array}{cc|c} \varepsilon_T & 0 & -ia \\ 0 & \varepsilon_T & -b \\ \hline ia & -b & \varepsilon_z\end{array}\right), \tag{92}$$

where the tensors are assumed to be constant at all positive frequencies ($\omega > 0$) with the parameters $\varepsilon_T = 2$, $\varepsilon_z = 1.5$, $a = -0.15$, and $b = 0.05$. This material can form a synthetic non-Abelian magnetic field $\mathcal{B} = 2k_0^2 ab\hat{\boldsymbol{e}}_z\hat{\sigma}_3$ acting on 2D optical waves, where the EM wave is described by a two-component wave function $\psi = (E_z, \eta_0 H_z)^{\mathsf{T}}$, with $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ denoting the impedance of free space [24]. As we have discussed, although the constitutive tensors are constant at positive frequencies, the negative frequency components are the complex conjugate of the positive ones; therefore, the material is generically dispersive, and the usual expression of the E-M tensor for real EM fields in nondispersive materials is invalid in this case. Nevertheless, for analytic signals, since only positive frequency components need to be taken into account, the dispersion-induced term vanishes in the Minkowski tensor Eq. (64).

For a given wave vector $\mathbf{k} = (k_x, k_y)$, there are two branches of eigenstates ψ_a $(a = 1, 2)$ satisfying

$$\psi_a = \frac{1}{\sqrt{h_0^2 + h_1^2 + h_2^2}} \begin{pmatrix} -h_0 \\ h_1 + i h_2 \end{pmatrix}, \quad (93)$$

with

$$h_0 = \frac{1}{\varepsilon_T}\left[k^2 + \frac{\omega_a^2}{c^2}(a^2 + b^2 - \varepsilon_z \varepsilon_T)\right],$$

$$h_1 = -\frac{\omega_a b k_x}{c \varepsilon_T},$$

$$h_2 = -\frac{\omega_a a k_y}{c \varepsilon_T}, \quad (94)$$

and the two corresponding eigen-frequencies

$$\omega_a(k_x, k_y) = \frac{c}{a^2 + b^2 - \varepsilon_z \varepsilon_T}[(k_x^2 - k_y^2)(b^2 - a^2) + k^2 \varepsilon_z \varepsilon_T + (-1)^a \left(k_x^2 b^2 + k_y^2 a^2\right)^{1/2} \left(k^2 \varepsilon_T \varepsilon_z - k_y^2 b^2 - k_x^2 a^2\right)^{1/2}]^{1/2}. \quad (95)$$

We consider a quasi-monochromatic 2D wave packet propagating along the x direction with a fixed k_x component of the wave vector while being finite in the y direction:

$$\begin{pmatrix} E_z \\ \eta_0 H_z \end{pmatrix} = e^{i k_x x} \int_{-\infty}^{\infty} dk_y \, e^{i k_y y} \sum_{a=1,2} f_a(k_y) \psi_a(k_y) e^{-i\omega_a(k_y) t}. \quad (96)$$

As k_x is fixed, ω_a, ψ_a are functions depending only on k_y. Here, we let the wave packet have a Gaussian-type initial profile with inverse width δ_y:

$$\psi(0, y) = \int_{-\infty}^{\infty} dk_y \, e^{i k_y y} e^{-(k_y/\delta_y)^2} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \quad (97)$$

Then, the superposition coefficients $f_a(k_y)$ as well as the evolution of the wave are determined by the initial condition. We require that the initial field contains components on both branches, i.e., $c_1, c_2 \neq 0$, which is demonstrated as a necessary condition to achieve trembling motion [24].

In Fig. 2(a), we plot the energy density of the analytic signal Eq. (65), which is equivalent to the time-period-averaged energy density of the quasi-monochromatic wave. It clearly shows that the intensity centroid oscillates with time in the y direction, transverse to beam propagation direction. The y components of Minkowski and Abraham momentum densities for the analytic signal are shown in Figs. 2(b) and 2(c), respectively. The Minkowski momentum density takes opposite signs in the upper and lower halves. In contrast, the Abraham momentum density distribution indicates that the fields move in a uniform

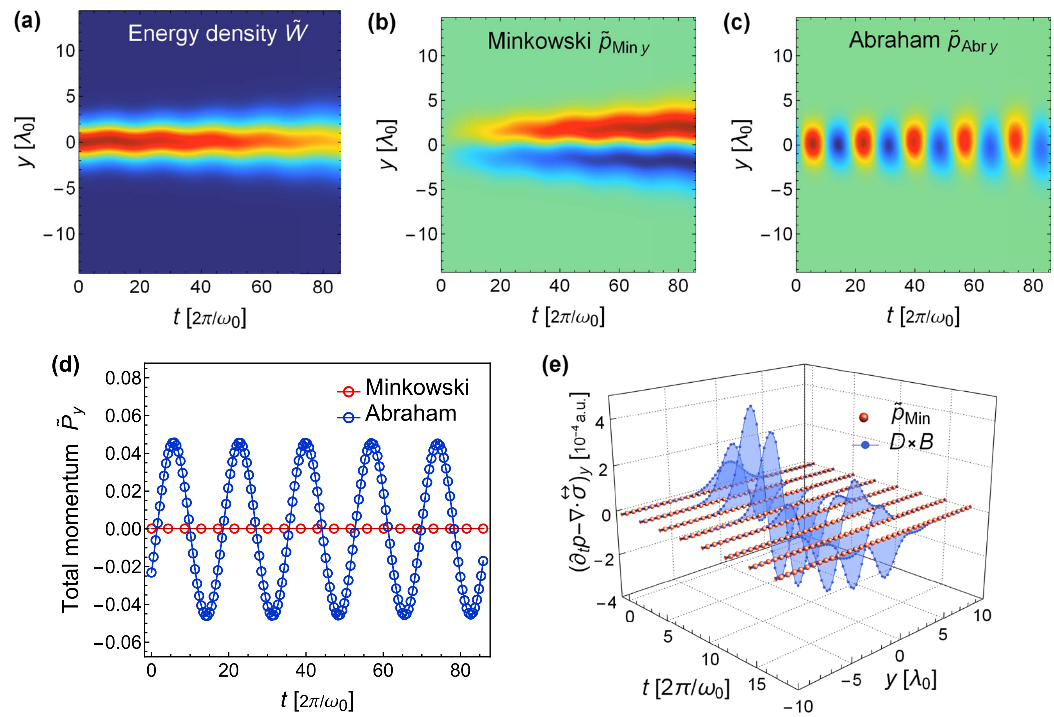

Fig. 2. Electromagnetic momenta in the Zitterbewegung effect of a Gaussian-type wave packet. (a) Distributions of the energy density of analytic signal (i.e., quasi-time-period value of the real field) evolving with time. (b), (c) Distribution and evolution of the y component of (b) Minkowski momentum density $\tilde{\mathbf{p}}_{\text{Min}} = \text{Re}(\mathbf{D}^* \times \mathbf{B})$ and of (c) Abraham momentum density $\tilde{\mathbf{p}}_{\text{Abr}} = \mathbf{S}/c^2 = \text{Re}(\mathbf{E}^* \times \mathbf{H})$. (d) Total momenta along y direction of the two types of definitions change with time. (f) Test of local momentum conservation, $\partial_t \mathbf{p} - \nabla \cdot \overleftrightarrow{\sigma} \stackrel{?}{=} 0$, for the conventional definition of Minkowski momentum $\mathbf{p} = \mathbf{D} \times B$ associated with real EM fields (blue dotted line) and Minkowski momentum $\tilde{\mathbf{p}}_{\text{Min}}$ [Eq. (67)] associated with the analytic signal (red spheres).

direction at each time point, while the direction alternates periodically in time. Also, in Fig. 2(d), the curves of the total y component of the two kinds of momenta demonstrate the conservatoin of total Minkowski momentum as a consequence of space translation symmetry, whereas the oscillation of the total Abraham momentum density provides a visual piece of evidence of the nonconservativeness of the Abraham momentum.

We compare the local momentum conservativeness of instantaneous real EM fields and the complex analytic signals (i.e., quasi-time-averaged fields) in this special kind of material:

(1) if the conventional expression of instantaneous Minkowski momentum density $\mathbf{D}\times\mathbf{B}$ can preserve the local momentum current in this complex-valued material: $\partial_t(\mathbf{D}\times\mathbf{B}) - \nabla\cdot\overleftrightarrow{\sigma}_{\mathrm{Min}} \overset{?}{=} 0$;

(2) if the modified Minkowski momentum density $\tilde{\mathbf{p}}_{\mathrm{Min}} = \mathrm{Re}[\mathbf{D}^*\times\mathbf{B}]$ for analytic signals can preserve the local momentum current: $\partial_t\tilde{\mathbf{p}}_{\mathrm{Min}} - \nabla\cdot\tilde{\overleftrightarrow{\sigma}}_{\mathrm{Min}} \overset{?}{=} 0$.

As the medium given by Eq. (91) possesses complex-valued components, the numerical computation verifies that only $\partial_t\tilde{\mathbf{p}}_{\mathrm{Min}} - \nabla\cdot\tilde{\overleftrightarrow{\sigma}}_{\mathrm{Min}} = 0$ associated with the analytic signals is satisfied, while $\partial_t(\mathbf{D}\times\mathbf{B}) - \nabla\cdot\overleftrightarrow{\sigma}^{(R)}_{\mathrm{Min}} \neq 0$ as shown in Fig. 2(e), indicating the modification induced by dispersion is necessary for instantaneous real fields.

7. CONCLUSION

To summarize, we have derived the EM E-M balance equation for instantaneous real EM fields in lossless bianisotropic media with frequency dispersion using Noether's theorem, and have obtained the generalized Minkowski and Abraham E-M tensors. The E-M conservation law for EM analytic signals is also derived explicitly for the first time, which proves successful to characterize the envelope evolution of time-averaged waves under quasi-monochromatic approximation. For both real fields and analytic signals, we show that the conserved EM quantities protected by the space translation symmetry of a homogeneous medium are the generalized Minkowski momenta (42) and (67), in which the the material dispersion contributes an additional term apart from the conventional term $\mathbf{D}\times\mathbf{B}$. On the contrary, the generalized Abraham momenta (80) and (88), which determine the centroid motion of wave packets, are generically nonconserved in bianisotropic media.

We stress that our results are obtained via regarding the media as external non-dynamic background of an open system, while the force and momentum transfer between light and surrounding material, treating them as a united closed system, is out of our scope. Nevertheless, our framework is indeed closely related to many exotic optical effects in bianisotropic media. In the paper, we have used our results to explain the countourintuitive effect that light can propagate along curved trajectories even in homogeneous anisotropic materials [24], and the results can also be applied to the study of optical forces exerted by surface waves on the boundaries of dispersive bianisotropic media [48,49].

In future studies, several problems wait to be solved. The first problem is how to incorporate the dissipation of media into the macroscopic Lagrangian and variational formalism. As it was shown that the expression of a Minkowski E-M tensor derived from lossy microscopic models cannot be naively extended from the result of the conservative case [9,59], the generalization from macroscopic models is quite challenging. Another problem is the spin–orbital decomposition of the Minkowski momentum in general bianisotropic media. As shown in Eq. (61), if we directly follow the definition of the orbital and spin momenta in isotropic media [12,13], two additional complicated terms appear in the decomposition. Therefore, searching another decomposition with simple and clear physical meaning is still desirable. The third problem is to derive the EM angular momentum in dispersive bianisotropic media. Since anisotropy breaks rotation symmetry in general cases, a conserved angular momentum does not exist even in homogeneous anisotropic media [11]. Hence we expect that the conservation law and the spin–orbital decomposition of angular momentum in generic bianisotropic materials would be more complicated than in isotropic media [13,60].

APPENDIX A: DERIVATION OF EQ. (53)

With replacing $\partial_j \to ik_j$ into Eq. (52), we have

$$\begin{aligned}\langle\mathbf{p}_{\mathrm{Min}}\rangle &= \frac{1}{2}\mathrm{Re}(\tilde{\mathbf{D}}^*\times\tilde{\mathbf{B}}) + \frac{1}{4}\mathrm{Re}\left(\tilde{\Psi}^\dagger\hat{\sigma}_3\frac{\partial\overleftrightarrow{N}(\omega)}{\partial\omega}\tilde{\Psi}\right)\mathbf{k}\\ &= \frac{1}{2}\mathrm{Re}(\tilde{\mathbf{D}}^*\times\tilde{\mathbf{B}}) + \frac{1}{4}\left(\tilde{\Phi}^\dagger\frac{\partial\overleftrightarrow{M}(\omega)}{\partial\omega}\tilde{\Phi}\right)\mathbf{k}.\end{aligned} \quad \text{(A1)}$$

On the other hand, the Maxwell's equation for plane waves reads

$$\begin{pmatrix} 0 & -\mathbf{k}\times \\ \mathbf{k}\times & 0 \end{pmatrix}\tilde{\Phi} = \omega\overleftrightarrow{M}(\omega)\tilde{\Phi}. \quad \text{(A2)}$$

Left multiplication of the Maxwell's equation by $\tilde{\Phi}^\dagger$ yields

$$\begin{aligned}\tilde{\Phi}^\dagger\overleftrightarrow{M}(\omega)\tilde{\Phi} &= \tilde{\Phi}^\dagger\begin{pmatrix} 0 & -\mathbf{k}\times \\ \mathbf{k}\times & 0 \end{pmatrix}\tilde{\Phi}\\ &= \tilde{\mathbf{H}}^*\cdot(\mathbf{k}\times\tilde{\mathbf{E}}) - \tilde{\mathbf{E}}^*\cdot(\mathbf{k}\times\tilde{\mathbf{H}})\\ &= \tilde{\mathbf{H}}^*\cdot(\mathbf{k}\times\tilde{\mathbf{E}}) - \tilde{\mathbf{E}}^*\cdot(\mathbf{k}\times\tilde{\mathbf{H}})\\ &= 2\mathrm{Re}\left(\tilde{\mathbf{E}}\cdot(\mathbf{k}\times\tilde{\mathbf{H}}^*) + \underbrace{\mathbf{k}\cdot(\mathbf{k}\times\tilde{\mathbf{H}}^*)}_{=0}\right)\\ &= -2\mathrm{Re}((\mathbf{k}\times\tilde{\mathbf{H}}^*)\times(\mathbf{k}\times\tilde{E})) = 2\omega^2\mathrm{Re}(\tilde{\mathbf{D}}^*\times\tilde{\mathbf{B}}).\end{aligned} \quad \text{(A3)}$$

Therefore, plugging $\mathrm{Re}(\tilde{\mathbf{D}}^*\times\tilde{\mathbf{B}}) = \frac{1}{2\omega}\tilde{\Phi}^\dagger\overleftrightarrow{M}(\omega)\tilde{\Phi}$ into Eq. (A1), we obtain Eq. (53) in the main text.

APPENDIX B: DERIVIATION OF EQ. (55)

The derivative of the Maxwell's equation (A2) with respect to $\mathbf{k}$ generates

$$\begin{pmatrix} 0 & -\overleftrightarrow{I}\times \\ \overleftrightarrow{I}\times & 0 \end{pmatrix}\tilde{\Phi} + \begin{pmatrix} 0 & -\mathbf{k}\times \\ \mathbf{k}\times & 0 \end{pmatrix}(\nabla_{\mathbf{k}}\tilde{\Phi})$$
$$= \left(\nabla_{\mathbf{k}}(\omega\overleftrightarrow{M}(\omega))\right)\tilde{\Phi} + \omega\overleftrightarrow{M}(\nabla_{\mathbf{k}}\tilde{\Phi}). \quad \textbf{(B1)}$$

Multiplying $\tilde{\Phi}^{\dagger}$ from the left side of the above equation, we get

$$\tilde{\Phi}^{\dagger}\begin{pmatrix} 0 & -\overleftrightarrow{I}\times \\ \overleftrightarrow{I}\times & 0 \end{pmatrix}\tilde{\Phi} - (\nabla_{\mathbf{k}}\omega)\tilde{\Phi}^{\dagger}\frac{\partial(\omega\overleftrightarrow{M}(\omega))}{\partial\omega}\tilde{\Phi}$$
$$= \left[\omega\tilde{\Phi}^{\dagger}\overleftrightarrow{M} - \tilde{\Phi}^{\dagger}\begin{pmatrix} 0 & -\mathbf{k}\times \\ \mathbf{k}\times & 0 \end{pmatrix}\right](\nabla_{\mathbf{k}}\tilde{\Phi}) = 0. \quad \textbf{(B2)}$$

And since

$$\tilde{\Phi}^{\dagger}\begin{pmatrix} 0 & -\overleftrightarrow{I}\times \\ \overleftrightarrow{I}\times & 0 \end{pmatrix}\tilde{\Phi} = 2\mathrm{Re}(\mathbf{E}^{*}\times\mathbf{H}), \quad \textbf{(B3)}$$

we obtain

$$\nabla_{\mathbf{k}}\omega = \frac{2\mathrm{Re}(\mathbf{E}^{*}\times\mathbf{H})}{\tilde{\Phi}^{\dagger}\frac{\partial(\omega\overleftrightarrow{M}(\omega))}{\partial\omega}\tilde{\Phi}} = \frac{\langle\mathbf{S}\rangle}{\langle W\rangle}. \quad \textbf{(B4)}$$

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